Nonradial stability of marginal stable circular orbits in stationary axisymmetric spacetimes

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We study linear nonradial perturbations and stability of a marginal stable circular orbit such as the innermost stable circular orbit of a test particle in stationary axisymmetric spacetimes which possess a reflection symmetry with respect to the equatorial plane. A zenithal stability criterion is obtained in terms of the metric components, the specific energy, and angular momentum of a test particle. The proposed approach is applied to the Kerr solution and Majumdar-Papapetrou solution to the Einstein equation. Moreover, we reexamine marginal stable circular orbits for a modified metric of a rapidly spinning black hole that has been recently proposed by Johannsen and Psaltis [Phys. Rev. D 83, 124015 (2011)]. We show that, for the Johannsen and Psaltis model, circular orbits that are stable against radial perturbations for some parameter region become unstable against zenithal perturbations. This suggests that the last circular orbit for this model may be larger than the innermost stable circular orbit.

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**Introduction**

![Graph showing effective potential $V_{\text{eff}}$ vs. $r/m$ for different values of angular momentum $L$ in both Newtonian and General Relativity contexts.](image)

- **Newtonian**

- **General Relativity** (Schwarzschild spacetime)
  - $L$: Angular momentum per unit mass
  - $L = 2$
  - $L = \sqrt{3}$
  - $L = 1$

- **Innermost stable circular orbit (ISCO)**
Introduction

ISCOs may be the transition points from the inspiral phase to the merging one.

s.t. Last circular orbits

[L. Blanchet, Living Rev. Relativ. 9, 4 (2006);
Stute and Camenzind’s equation for the MSCO radius in general form of a stationary axisymmetric spacetime

\[ ds^2 = f(dt - wd\phi)^2 - f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] \]

\[
\left[ w_{,\rho} w_{,\rho \rho} f^5 \rho (2f - f_{,\rho \rho}) + w_{,\rho} f^4 [2f^2 + (-f_{,\rho} + f_{,\rho \rho} f) \rho^2] + w_{,\rho} f^2 \sqrt{w_{,\rho} f^4 + f_{,\rho} \rho (2f - f_{,\rho} \rho)} \times [2f^2 + 2f_{,\rho} \rho^2 - f\rho (4f_{,\rho} + f_{,\rho \rho} \rho)] + \rho (2f - f_{,\rho} \rho) \{3f_{,\rho} f^2 - 4f_{,\rho} f \rho + f_{,\rho}^3 \rho^2 + f^2 [f_{,\rho \rho} \rho - w_{,\rho \rho} f \sqrt{w_{,\rho} f^4 + f_{,\rho} \rho (2f - f_{,\rho} \rho)}] \} / \left[ f^2 \rho^2 \{w_{,\rho} f^4 + f_{,\rho} \rho f - f_{,\rho}^2 \rho^2 - f^2 [2 + w_{,\rho} \sqrt{w_{,\rho} f^4 + f_{,\rho} \rho (2f - f_{,\rho} \rho)}] \} \right] \right] = 0. \]
Marginal stable circular orbits

This work

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- Stationary and axisymmetric spacetime
- **Nonradial stability criterion** is NEW

![Diagram showing radial stability and nonradial stability](image-url)
Marginal stable circular orbits
c
consider a stationary axisymmetric spacetime


\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu
\]

\[
= - A(y^p, y^q) dt^2 - 2H(y^p, y^q) dt d\phi
\]

\[
+ F(y^p, y^q) (\gamma_{pq} dy^p dy^q) + D(y^p, y^q) d\phi^2
\]

choose the polar coordinates

\[
ds^2 = - A(r, \theta) dt^2 - 2H(r, \theta) dt d\phi + B(r, \theta) dr^2
\]

\[
+ C(r, \theta) d\theta^2 + D(r, \theta) d\phi^2
\]

assume a local reflection symmetry with \( \theta = \frac{\pi}{2} \)

\[
\left. \frac{\partial g_{\mu\nu}}{\partial \theta} \right|_{\theta = \pi/2} = 0
\]
Marginal stable circular orbits

contravariant metric tensor

\[
g^{\mu\nu} = \begin{pmatrix}
-\frac{D}{AD+H^2} & 0 & 0 & -\frac{H}{AD+H^2} \\
0 & \frac{1}{B} & 0 & 0 \\
0 & 0 & \frac{1}{C} & 0 \\
-\frac{H}{AD+H^2} & 0 & 0 & \frac{A}{AD+H^2}
\end{pmatrix} = \begin{pmatrix}
-\tilde{A} & 0 & 0 & -\tilde{H} \\
0 & \tilde{B} & 0 & 0 \\
0 & 0 & \tilde{C} & 0 \\
-\tilde{H} & 0 & 0 & \tilde{D}
\end{pmatrix}
\]

Lagrangian of the test particle

\[
\mathcal{L} = -A\dot{t}^2 - 2Ht\dot{\phi} + B\dot{r}^2 + C\dot{\theta}^2 + D\dot{\phi}^2
\]

two constants of motion

\[
\varepsilon \equiv \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \ddot{t}} = -At - H\dot{\phi} \quad , \quad l \equiv \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -Ht + D\dot{\phi}
\]
Marginal stable circular orbits (radial stability)

\[ r \text{-component of the geodesic equation} \]

\[
\ddot{r} + \frac{1}{2B(r, \theta)} \left\{ \frac{\partial B(r, \theta)}{\partial r} \dot{r}^2 - \frac{\partial C(r, \theta)}{\partial r} \dot{\theta}^2 + 2 \frac{\partial B(r, \theta)}{\partial \theta} \dot{r} \dot{\theta} \right\} \\
= -\frac{1}{2B(r, \theta)} \left\{ \frac{\partial A(r, \theta)}{\partial r} \dot{t}^2 + 2 \frac{\partial H(r, \theta)}{\partial r} \dot{t} \dot{\phi} - \frac{\partial D(r, \theta)}{\partial r} \dot{\phi}^2 \right\}
\]

substitute \( \dot{t} = -\tilde{A} \varepsilon - \tilde{H} l, \quad \dot{\phi} = -\tilde{H} \varepsilon + \tilde{D} l \)

\[
2B \ddot{r} + \frac{\partial B}{\partial r} \dot{r}^2 - \frac{\partial C}{\partial r} \dot{\theta}^2 + 2 \frac{\partial B}{\partial \theta} \dot{r} \dot{\theta} = \frac{\partial \tilde{A}}{\partial r} \varepsilon^2 + 2 \frac{\partial \tilde{H}}{\partial r} \varepsilon l - \frac{\partial \tilde{D}}{\partial r} l^2
\]
Marginal stable circular orbits (radial stability)

perturbed position \( r = r_c + \delta r, \quad \theta = \frac{\pi}{2} + \delta \theta \)

the Taylor series approximation

\[
\begin{align*}
2 \left( B|_c + \frac{\partial B}{\partial r}|_c \delta r + \frac{\partial B}{\partial \theta}|_c \delta \theta \right) \delta r + \left( \frac{\partial B}{\partial r}|_c + \frac{\partial^2 B}{\partial r^2}|_c \delta r + \frac{\partial^2 B}{\partial r \partial \theta}|_c \delta \theta \right) \delta r^2 \\
- \left( \frac{\partial C}{\partial r}|_c + \frac{\partial^2 C}{\partial r^2}|_c \delta r + \frac{\partial^2 C}{\partial r \partial \theta}|_c \delta \theta \right) \delta \theta^2 + 2 \left( \frac{\partial B}{\partial \theta}|_c + \frac{\partial^2 B}{\partial r \partial \theta}|_c \delta r + \frac{\partial^2 B}{\partial \theta^2}|_c \delta \theta \right) \delta r \delta \theta \\
= \left( \frac{\partial \tilde{A}}{\partial r}|_c + \frac{\partial^2 \tilde{A}}{\partial r^2}|_c \delta r + \frac{\partial^2 \tilde{A}}{\partial r \partial \theta}|_c \delta \theta \right) \varepsilon^2 + 2 \left( \frac{\partial \tilde{H}}{\partial r}|_c + \frac{\partial^2 \tilde{H}}{\partial r^2}|_c \delta r + \frac{\partial^2 \tilde{H}}{\partial r \partial \theta}|_c \delta \theta \right) \varepsilon l \\
- \left( \frac{\partial \tilde{D}}{\partial r}|_c + \frac{\partial^2 \tilde{D}}{\partial r^2}|_c \delta r + \frac{\partial^2 \tilde{D}}{\partial r \partial \theta}|_c \delta \theta \right) l^2
\end{align*}
\]

zeroth order

\[
\left. \frac{\partial \tilde{A}}{\partial r} \right|_c \varepsilon^2 + 2 \left. \frac{\partial \tilde{H}}{\partial r} \right|_c \varepsilon l - \left. \frac{\partial \tilde{D}}{\partial r} \right|_c l^2 = 0
\]

linear order

\[
\left. \frac{\partial^2 \tilde{A}}{\partial r^2} \right|_c \varepsilon^2 + 2 \left. \frac{\partial^2 \tilde{H}}{\partial r^2} \right|_c \varepsilon l - \left. \frac{\partial^2 \tilde{D}}{\partial r^2} \right|_c l^2 = 0
\]
Marginal stable circular orbits (radial stability)

we obtain

$$\left[ \frac{d\tilde{A}}{dr} \frac{d^2 \tilde{D}}{dr^2} - \frac{d\tilde{D}}{dr} \frac{d^2 \tilde{A}}{dr^2} \right]^2$$

$$- 4 \left[ \frac{d\tilde{A}}{dr} \frac{d^2 \tilde{H}}{dr^2} - \frac{d\tilde{H}}{dr} \frac{d^2 \tilde{A}}{dr^2} \right] \left[ \frac{d\tilde{D}}{dr} \frac{d^2 \tilde{H}}{dr^2} - \frac{d\tilde{H}}{dr} \frac{d^2 \tilde{D}}{dr^2} \right] = 0$$

This agrees with the Stute and Camenzind’s equation for the MSCO radius

[M. Stute and M. Camenzind, Mon. Not. R. Soc. 336, 831 (2002)]

We call this equation MSCO equation.
Marginal stable circular orbits (nonradial stability)

$\theta$-component of the geodesic equation

linear order

$$2C \left( r_c, \frac{\pi}{2} \right) \frac{\ddot{\delta \theta}}{\delta \theta} = \frac{\partial^2 \tilde{A}}{\partial \theta^2} \bigg|_c \epsilon^2 + 2 \frac{\partial^2 \tilde{H}}{\partial \theta^2} \bigg|_c \epsilon l - \frac{\partial^2 \tilde{D}}{\partial \theta^2} \bigg|_c l^2$$

$$\frac{\ddot{\delta \theta}}{\delta \theta} < 0 \iff \frac{\partial^2 \tilde{A}}{\partial \theta^2} \bigg|_c \epsilon^2 + 2 \frac{\partial^2 \tilde{H}}{\partial \theta^2} \bigg|_c \epsilon l - \frac{\partial^2 \tilde{D}}{\partial \theta^2} \bigg|_c l^2 < 0 \quad \text{stable}$$

$$\frac{\ddot{\delta \theta}}{\delta \theta} > 0 \iff \frac{\partial^2 \tilde{A}}{\partial \theta^2} \bigg|_c \epsilon^2 + 2 \frac{\partial^2 \tilde{H}}{\partial \theta^2} \bigg|_c \epsilon l - \frac{\partial^2 \tilde{D}}{\partial \theta^2} \bigg|_c l^2 > 0 \quad \text{unstable}$$
Johannsen and Psaltis’s model

Johannsen and Psaltis’s modified Kerr metric as

\[ ds^2 = - \left[ 1 + h(r, \theta) \right] \left( 1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4aMr \sin^2 \theta}{\Sigma} \times [1 + h(r, \theta)] dt d\phi \]

\[ + \frac{\Sigma[1 + h(r, \theta)]}{\Delta + a^2 \sin^2 \theta h(r, \theta)} dr^2 + \Sigma d\theta^2 + \left[ \sin^2 \theta \left( r^2 + a^2 + \frac{2a^2 Mr \sin^2 \theta}{\Sigma} \right) + h(r, \theta) \frac{a^2(\Sigma + 2Mr) \sin^4 \theta}{\Sigma} \right] d\phi^2 \]

\[ \Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \]

\[ h(r, \theta) = \epsilon_3 \frac{M^3r}{\Sigma^2} \]


The new parameter \( \epsilon_3 \) is not constrained by observations so far.
Johannsen and Psaltis’s model

$\epsilon_3 = 2$ case

prograde case

retrograde case

root of MSCO eq.
Johannsen and Psaltis’s model

$\epsilon_3 = 2$ case

prograde case

 ISC0 radius

 Last circular orbit

$\frac{r}{m}$

$\frac{a}{m}$
Conclusion

We performed the linear stability analysis to nonradial perturbations of a marginal stable circular orbit of a test particle in stationary axisymmetric spacetimes.

The nonradial stability criteria was obtained as

\[
\left. \frac{\partial^2 \tilde{A}}{\partial \theta^2} \right|_c \varepsilon^2 + 2 \left. \frac{\partial^2 \tilde{H}}{\partial \theta^2} \right|_c \varepsilon l - \left. \frac{\partial^2 \tilde{D}}{\partial \theta^2} \right|_c l^2 < 0
\]

The proposed approach was applied to the Johannsen and Psaltis’s model

At least for the Johannsen and Psaltis’s model, the last circular orbit might be larger than the ISCO.
Future work

another examples:
Kerr-de Sitter spacetime, Kerr-Newman spacetime
zenithal stability
extension to a more general case:
In this work, orbital plane is limited to
equatorial plane → arbitrary plane