Perturbative uniqueness of a static photon surface

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arXiv:1607.07133

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Photon sphere

What characterize a black hole?

- Existence of a horizon
- Existence of circular orbits of null geodesics

In a Schwarzschild spacetime, collection of circular orbits of null geodesics form a timelike surface:

**Photon sphere** \( r = 3M \)
Photon surface (i)

Claudel, Virbhadra & Ellis, JMP 42 (2001) 818 [gr-qc/0005050]

Proposed as generalization of a photon sphere

If there is a timelike hypersurface $S$ such that any null geodesics initially tangent to $S$ continue to be included in $S$, we call $S$ a photon surface.

- not necessarily static,
- not necessarily spherically symmetric.
Photon surface (2)

Example of photon surfaces:

- \((r=3M) \times (\text{time})\) in a Schwarzschild spacetime
- \((\text{Plane}) \times (\text{time})\) in a flat space
- Hyperbolic surface in a flat spacetime

The followings are not photon surfaces

- Light rings in a Kerr spacetime
- Event horizons
Problem

Can a static distorted photon surface exist?

Asymptotically flat

Static, Vacuum

\[ ds^2 = -N^2 dt^2 + g_{ij} dx^i dx^j \]

\[ G_{\mu\nu} = 0 \]

We do not care about inner region
Partial answer

Uniqueness of a (redefined) photon sphere

Cederbaum, arXiv:1406.5475

Photon surface + $N=\text{const.}$

The spacetime is a Schwarzschild spacetime.

Uniqueness theorems of a (redefined) photon sphere

Vacuum

Cederbaum, arXiv:1406.5475
Cederbaum & Galloway, arXiv:1504.05804

Electrovacuum

Yazadjiev & Lazov, CQG32 (2015) 165021
Cederbaum & Galloway, CQG33 (2015) 075006

Others

Yazadjiev, PRD91 (2015) 123013
Rogatko, PRD93 (2016) 064003
What we do is...

- The assumption of the constancy of the lapse on a redefined photon sphere is strong.

- We would like to know whether
  - the uniqueness theorem can be proved without assuming constancy of lapse, or
  - a distorted photon surface can be present if we omit the assumption of constancy of lapse

- We study “Perturbative uniqueness” of a static photon surface.
  - We consider static perturbation of a Schwarzschild spacetime, and
  - study whether a photon surface can exist in a distorted spacetime.

C.f. Perturbative uniqueness of a higher-dimensional static vacuum spacetime

Kodama, PTP112 (2004) 249
Photon surface condition

Claudel, Virbhadra & Ellis, JMP42 (2001) 818 [gr-qc/0005050]

- **S** is a Photon surface

- Any null geodesics of a hypersurface \( S \) is simultaneously a null geodesic of spacetime \( \mathcal{M} \)

- For arbitrary null tangent vector \( k^a \) of \( S \),
  \[ \chi_{ab} k^a k^b = 0 \]
  holds.

- **S** is umbilical:
  \[ \chi_{ab} \propto h_{ab} \]

- **Photon surface condition**

\[ S = \int \mathcal{L} d\lambda, \quad \mathcal{L} = \frac{1}{2} g_{ab} k^a k^b. \]
Static perturbation of a Schwarzschild spacetime \((\mathbf{1})\)

- **Background spacetime**

\[
\begin{aligned}
\hat{ds}^2 &= -e^{2\nu(0)} dt^2 + e^{2\mu(0)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \\
\nu(0) &= e^{-2\mu(0)} = 1 - \frac{2M}{r}.
\end{aligned}
\]

- **Spacetime distorted by static even-parity perturbation (in Regge-Wheeler gauge) \((\ell \geq 2)\):**

\[
\begin{aligned}
\hat{ds}^2 &= -e^{2\nu} dt^2 + e^{2\mu} dr^2 + e^{2\psi} r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \\
\nu &= \nu(0) + \epsilon \nu(1) + \cdots, \\
\mu &= \mu(0) + \epsilon \mu(1) + \cdots, \\
\psi &= \epsilon \psi(1) + \cdots,
\end{aligned}
\]

\[
\begin{aligned}
\nu(1) &= -\mu(1) = -H^{(1)}(r)Y_{\ell m}(\theta, \phi), \\
\psi(1) &= K^{(1)}(r)Y_{\ell m}(\theta, \phi),
\end{aligned}
\]

- **Analytic solution:**

\[
H^{(1)}(x) = \alpha_\ell P^2_\ell(2x - 1) + \beta_\ell Q^2_\ell(2x - 1),
\]

- **Behavior:**

- Regular on the horizon,
- Divergent at infinity.
- Divergent on the horizon,
- Decays at infinity.
Photon surface condition (1)

Position of a photon surface: \( r = f(\theta, \phi) \),

\[ f = f^{(0)} + \epsilon f^{(1)} + \cdots, \quad \text{with} \quad f^{(0)} = 3M. \]

Photon surface condition \( \chi_{ab} \propto h_{ab} \)

- \( f_{,\theta\phi}^{(1)} = \cot \theta f_{,\phi}^{(1)} \),
- \( f_{,\phi\phi}^{(1)} = \sin^2 \theta f_{,\theta\theta}^{(1)} - \sin \theta \cos \theta f_{,\theta}^{(1)} \),
- \( f^{(1)} = \sin \theta (\alpha e^{i\phi} + \beta e^{-i\phi}) + \gamma \cos \theta + \delta, \)
- \( f^{(1)} = 0. \)

In the Regge-Wheeler gauge, a photon surface (if it exists) remains at the same coordinate position.
Photon surface condition (2)

- Condition for the metric perturbation
  \[
  (\nu_r^{(1)} - \psi_r^{(1)}) \bigg|_{r=3M} = 0 \quad \iff \quad \frac{H^{(1)}_{,\tilde{x}}}{H^{(1)}} \bigg|_{\tilde{x}=2} = -\frac{1}{3}, \quad (\tilde{x} = r/M - 1)
  \]

- If the outside region is vacuum,
  \[
  H^{(1)}(x) = \alpha \ell P^2_\ell(2x - 1) + \beta \ell Q^2_\ell(2x - 1),
  \]
  \[
  \frac{d}{d\tilde{x}} \frac{Q^2_\ell(\tilde{x})}{Q^2_\ell(\tilde{x})} \bigg|_{\tilde{x}=2} = -\frac{3\ell + 1}{6} - \frac{1}{4} \left[ \frac{d}{dz} \frac{2F_1(a, b; c; z)}{2F_1(a, b; c; z)} \right]_{z=1/4} < -\frac{7}{6}
  \]
  \[
  a = \frac{\ell + 3}{2}, \quad b = \frac{\ell}{2} + 2, \quad c = \ell + \frac{3}{2}.
  \]

- Any static perturbation does not satisfy the photon surface condition (a photon surface vanishes once the spacetime is distorted)
If there is matter in the outside region

\[ H^{(1)}(x) = \alpha_\ell P^2_\ell (2x - 1) + \beta_\ell Q^2_\ell (2x - 1), \]

\[ \left. \frac{H^{(1)}(\bar{x})}{H^{(1)}} \right|_{\bar{x}=2} = \frac{1}{3}, \quad \text{Equation for } \alpha_\ell / \beta_\ell \]

<table>
<thead>
<tr>
<th>(\ell)</th>
<th>(\alpha_\ell / \beta_\ell) (numerical value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(\frac{26}{45} - \text{arctanh} \left(\frac{1}{2}\right)) (2.847 \times 10^{-2})</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{323}{685} - \text{arctanh} \left(\frac{1}{2}\right)) (1.121 \times 10^{-3})</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{511}{685} - \text{arctanh} \left(\frac{1}{2}\right)) (5.966 \times 10^{-5})</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{3722}{6772} - \text{arctanh} \left(\frac{1}{2}\right)) (3.564 \times 10^{-6})</td>
</tr>
</tbody>
</table>

If we fine tune the ratio of the amplitudes of two independent solutions, a distorted photon surface could be present.
Conclusion

- Perturbative Uniqueness Theorem
  A sequence of spacetime solutions possessing distorted photon surfaces that connects to the Schwarzschild solution does not exist if outside region is vacuum and asymptotically flat.

  - Distorted solution does not exist, or
  - Distorted solution exists but it does not connects to the Schwarzschild solution

- Conjecture
  Spacetime solution with a distorted photon surface does not exist if outside region is vacuum and asymptotically flat

- Possibility of distorted photon surface
  If we do not assume that outside region is vacuum, there is a possibility that a distorted photon surface be present.