Non-spherical gravitational collapse of a collisionless particle system

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△ Introduction

Motivation

1. Test of the cosmic censorship conjecture (CCC)
   - Growth of asphericity + Hoop conjecture → CCC violation?

2. Collapse in an early matter dominated era
   - E.g., Preheating era would be matter dominated → collapse with low pressure
   - Asphericity should be crucial
   - PBH?PNS (Primordial Naked Singularity)?

Purpose

- Full 3D relativistic simulation of spindle collapse of collisionless particles (without exact axisymmetry)
- Comparison with Shapiro and Teukolsky’s work (ST)
  [PRL 66, 994(1991)]
Review of Shapiro and Teukolsky’s results

Axial symmetric gravitational collapse

- Exactly axi-symmetric (2D simulation)
- Collisionless ring sources
- Spindle collapse without horizon

- The larger \( K_{\text{inv}} := R^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} \) for finer resolution
  \( \rightarrow \) naked singularity formation, violation of CCC

- Calculation breaks down because of the singularity
- The position of max \( K_{\text{inv}} \) is outside the matter
  - max \( K_{\text{inv}} \) on z-axis, \( t \sim 23M, z = 6.1M \)
  - density peak \( z \sim 5M \), matter edge \( z \sim 5.8M \)
△ Methods

[Shibata(gr-qc/9905058)]

◎ Geometrical variables

- Metric

\[ ds^2 = -\alpha^2 \, dt^2 + \gamma_{ij}(dx^i + \beta^i \, dt)(dx^j + \beta^j \, dt) \]

\[ \gamma_{ij} = e^{4\psi} \bar{\gamma}_{ij} \text{ with } \det \bar{\gamma} = 1 \]

- Projection tensor

\[ \gamma^\nu_\mu = n_\mu n^\nu + g^\nu_\mu \text{ with unit normal } \quad n_\mu := -\alpha (dt)_\mu \]

- Extrinsic curvature

\[ K_{ij} = -\gamma^\iota_\mu \gamma^\iota_j \nabla_\nu n_\nu = e^{4\psi} \widetilde{A}_{ij} + \frac{1}{3} K \gamma_{ij} \]

◎ Stress energy tensor

- For a point particle system

\[ E = n_\mu n_\nu T^{\mu \nu} = \sum_p m_p \Gamma_p \frac{\delta^3(\vec{x} - \vec{x}_p)}{\sqrt{\gamma}} \]

\[ j^i = -n_\nu \gamma^i_\mu T^{\mu \nu} = \sum_p m_p \Gamma_p V_p^i \frac{\delta^3(\vec{x} - \vec{x}_p)}{\sqrt{\gamma}} \]

\[ S^{ij} = \gamma^i_\mu \gamma^j_\nu T^{\mu \nu} = \sum_p m_p \Gamma_p V_p^i V_p^j \frac{\delta^3(\vec{x} - \vec{x}_p)}{\sqrt{\gamma}} \]

where particle 4-velocity

\[ u^\mu_p = \Gamma_p (n^\mu + V^\mu_p) \]

- Smoothing

\[ \delta^3(\vec{x} - \vec{x}_p) \]

\[ \rightarrow f_{sp}(|\vec{x} - \vec{x}_p|, r_s) \]
**Geodesic equations** [Vincent et al. (1208.3927)]

\[
\begin{align*}
\frac{dr_p}{dt} &= \frac{\alpha}{\Gamma_p}, \quad \frac{dx_p^i}{dt} = -\beta^i + \alpha V^i \\
\frac{d\Gamma_p}{dt} &= \Gamma_p V^j_p (\alpha K_{ij} V^j_p - \partial_i \alpha) \\
\frac{dv^i_p}{dt} &= \alpha V^j_p [V^i_p (\partial_j \ln \alpha - K_{jk} V^k_p) + 2K^i_j - V^k_p \Gamma^i_{jk}] - \gamma_{ij} \partial_j \alpha - V^i_p \partial_j \beta^i
\end{align*}
\]

**Cleaning of Hamiltonian constraint**

- Hamiltonian constraint as an elliptic equation

\[
\tilde{D}_i \tilde{D}^i \psi = -\tilde{D}_i \psi \tilde{D}^i \psi + \frac{1}{8} \tilde{R} + e^{4\psi} \left( \frac{1}{12} K^2 - \frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} - 2\pi E \right)
\]

A few steps of Successive-Over-Relaxation method to solve it at each time step to clean up errors

**Others**

- BSSN with 2nd order finite differences
- Maximal slice for \( \alpha \), hyperbolic gamma driver for \( \beta^i \)
- Kreiss-Oligar dissipation

**Calculation flow**

1. Evolve geometrical variables except for \( \alpha \)
2. Evolve particle variables solving the geodesic eqs.
3. Set energy momentum tensor
4. Clean the Hamiltonian constraint
5. Set \( \alpha \) by solving the elliptic eq. of maximal slice
**Assumptions**

- Conformally flat, momentarily static
  \[ \tilde{\gamma}_{ij} = \delta_{ij}, \quad K_{ij} = 0 \quad \Rightarrow \quad f^i = 0 \quad \Rightarrow \quad V^i_p = 0, \quad \Gamma_p = 1 \]
  momentum constraint

- Hamiltonian constraint
  \[ \Delta \Psi = -2\pi E\Psi^5 = -2\pi \sum_p f_{sp}(|\tilde{x} - \tilde{x}_p|, r_s)/\Psi \quad \text{where} \quad \Psi = e^\Phi \]
  numerically solved for given particle distribution

**Reference continuum (the same as ST)**

[Ref. Nakamura et. al (PRD38, 2972)]

- Energy density \( \bar{E} \) and the conformal factor \( \bar{\Psi} \)
  \[ \bar{E}\bar{\Psi}^5 = E_N = \frac{3M_N}{2\pi a^2 b} \quad \text{for} \quad \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leq 1 \]
  \[ = 0 \quad \text{for} \quad \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} > 1 \]

  \[ \Rightarrow \bar{\Psi} = 1 - \Phi \Rightarrow \text{Hamiltonian constraint:} \quad \Delta \Phi = 4\pi E_N \]
  \[ \Phi = \left[ \frac{3M_N}{4b^3 e^5} \left( \frac{b^2 e - \text{arccosh} \left( \frac{b}{c} \right) }{a^2} \right) \right] R^2 + \left[ \frac{3M_N}{2b^3 e^3} \left( -e + \text{arccosh} \left( \frac{c}{a} \right) \right) \right] z^2 - \frac{3}{2be} \text{arccosh} \left( \frac{c}{a} \right) \]
  where \( e = \sqrt{1 - a^2/b^2}, \quad R = \sqrt{x^2 + y^2} \)

- Mass
  \[ \lim_{r \to \infty} \bar{\Psi} = 1 - \lim_{r \to \infty} \Phi = 1 + \frac{M_N}{r} \Rightarrow \text{total mass:} \quad M = 2M_N \]
  rest mass: \( M_0 = \int \bar{E}\bar{\Psi}^6 d^3x = 2M_N + \frac{6}{5} M_N^2 \ln \frac{1+e}{1-e} \)

**Particle distribution**

- Number of particles \( \Delta N \) in a grid box \( \Delta V \)
  \[ \Delta N = \frac{\bar{E}\bar{\Psi}^6}{m} \Delta V = \frac{E_N \bar{\Psi}}{m} \Delta V \quad \text{with} \quad m = \frac{M_0}{N} \]
**Results**

**Parameters**

- **Numerical domain**
  
  \[ 0 < x, y, z < L \text{ with } L/M = 20 \]

- **Parameters (the same as ST)**
  
  \[ b/M = 10, e = 0.9 \]

- **Numerical parameters**
  
  \[ N = 5 \times 10^5, r_s = L/75, \text{ grid interval } \Delta = L/120 \]

**Constraint violation**

![Constraint violation graph]

Suppressing max norm of the momentum constraint is hard. We require at most \(~10\%\).

**Curvature invariants**

- **Kretschmann invariant:** \( K_{\text{inv}} = R^\mu_\nu\lambda_\sigma R_\mu\nu\lambda_\sigma \)

- **Weyl curvature invariant:** \( W_{\text{inv}} = C^\mu_\nu\lambda_\sigma C_\mu\nu\lambda_\sigma \)
Snapshots

## Density

- Kretschmann
- Weyl

**Density**

- **Kretschmann**
- **Weyl**

**Time:**
- $t=0$
- $t=23$
- $t=24.5$
- $t=26$

**Parameters:**
- $E \times 10^3$
- $K \times 10^7$
- $W \times 10$

**Variables:**
- $x$
- $y$
- $z$
**Time evolution of the peak values**

![Graph showing the time evolution of Kretschmann and Weyl squares](image)

**Resolution dependence of the max values**

![Graph showing the resolution dependence of Kretschmann and Weyl squares](image)

**Summary of results**

- Larger curvature invariants for finer resolution → naked singularity for infinite resolution
- Calculation does not break down even after the time of the maximum curvature invariant
- The position of the peak of the Kretschmann inv. is inside the matter distribution (Ricci part dominates). → structural instability of the ST type singularity?

**where** \( r_s \propto \Delta, \quad N \propto 1/\Delta^3 \)