# Time-delay inteferometry for LISA: The algebraic approach

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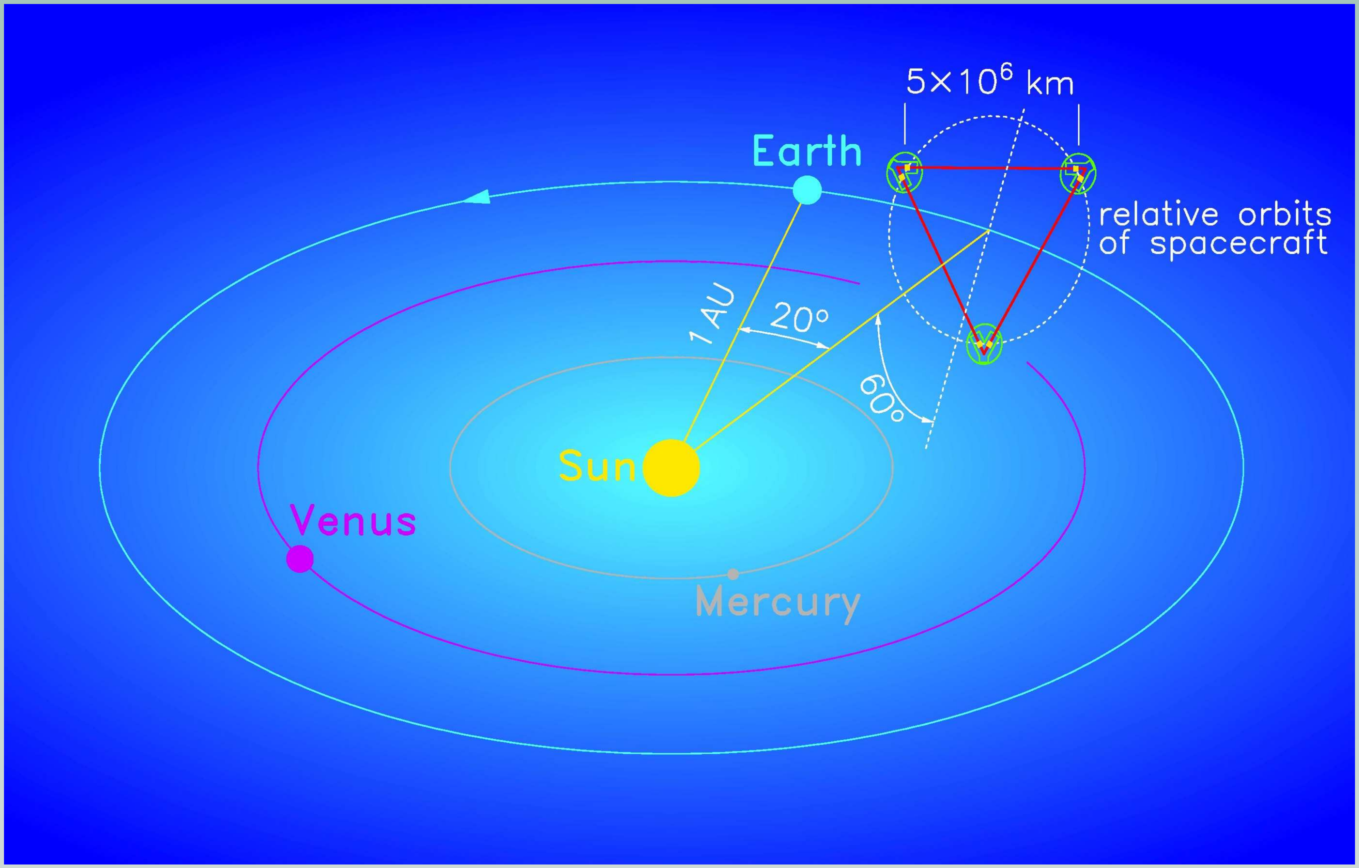
# Plan of Talk

- Introduction: LISA
- Laser frequency noise cancellation problem
- The TDI problem for LISA: Ist, modified Ist and IInd generation TDI
- Mathematical structure: module of syzygies
- Complete solution for first generation TDI: static LISA in flat spacetime
- General relativistic model of LISA: flexing arms
- Using symmetries to simplify IInd generation TDI
- Effect on TDI variables: Residual noise in modified first generation TDI
- IInd generation TDI solutions for LISA with one arm dysfunctional

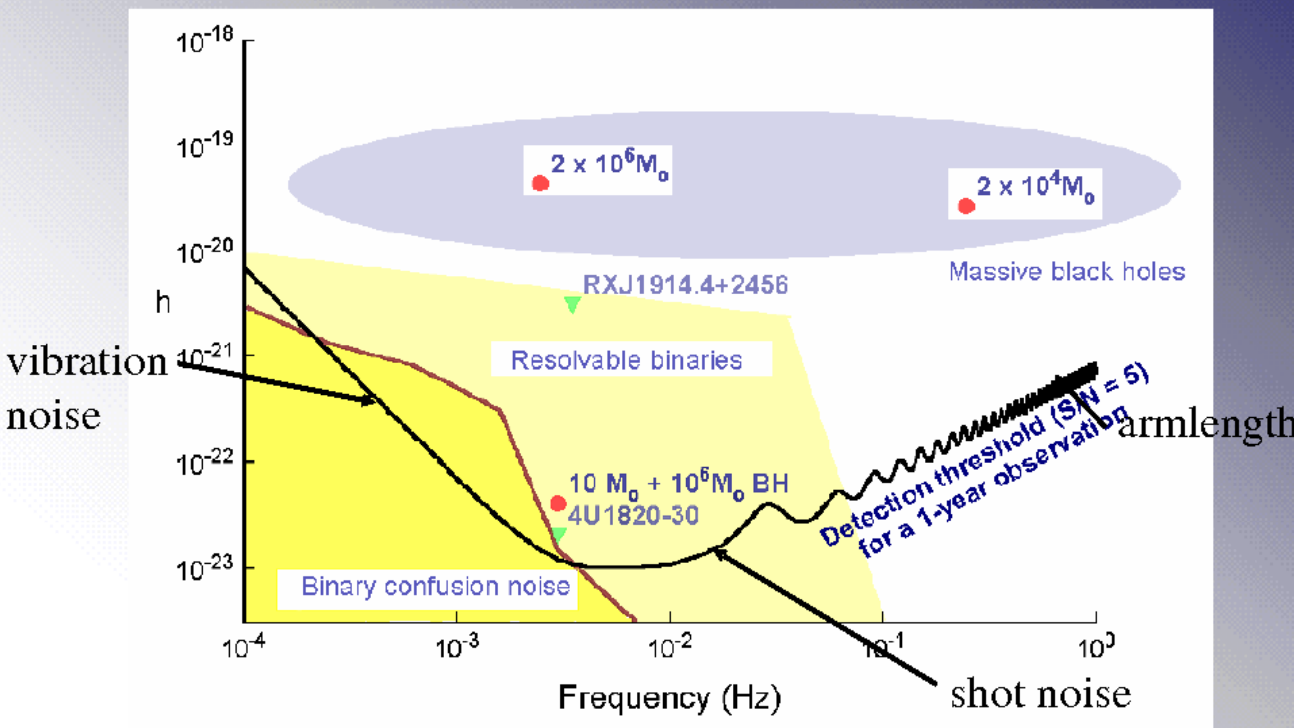
# Introduction: LISA

- Ground-based Laser Inteferometric Detector Network:
  LIGO, VIRGO, TAMA, GEO & AIGO - 10 Hz to KHz.
- Ground-based gravitational wave detectors are close to acquiring astrophysically interesting data - LIGO entering the advanced detector stage
- The Space Antenna LISA: Why go to Space?
  - Low frequency searches :$10^{-5}$ Hz to $10^{-1}$ Hz
    Complementary to groundbased detectors - just as the different astronomies, optical, radio, etc. complement each other.
  - Gauranteed Gravitational Wave Sources
- The LISA Project: ESA & NASA

# The LISA Project



# LISA noise curve



# Laser Frequency Noise

- Frequency of Laser: $\nu_0 \sim 3 \times 10^{14}$ Hz
- Frequency noise: $\widetilde{\Delta\nu} \sim 10$ Hz / $\sqrt{\mathrm{Hz}}$
- $h_{noise} = C(t) \equiv \frac{\Delta\nu(t)}{\nu_0} \sim 3 \times 10^{-14}$
- But $h_{sens} \sim 10^{-21}, 10^{-22}$
- **7 or 8 orders of magnitude**

# Cancelling laser frequency noise in an unequal arm Interferometer

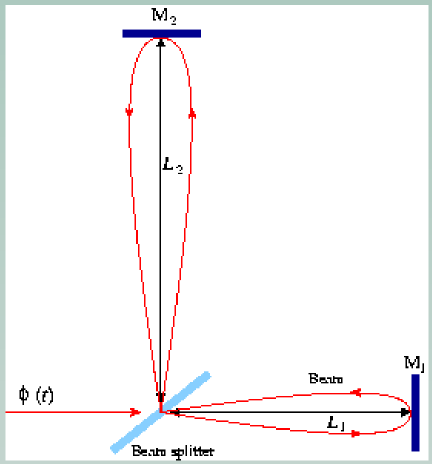
Equal arms: This noise is automatically cancelled

$C(t)$: Laser frequency noise

$$C_1(t) = C(t-2L_1) - C(t) = (\mathcal{D}_1^2 - 1)C(t)$$

$$C_2(t) = C(t-2L_2) - C(t) = (\mathcal{D}_2^2 - 1)C(t)$$

$$\Delta C(t) = C_1(t) - C_2(t) \neq 0$$

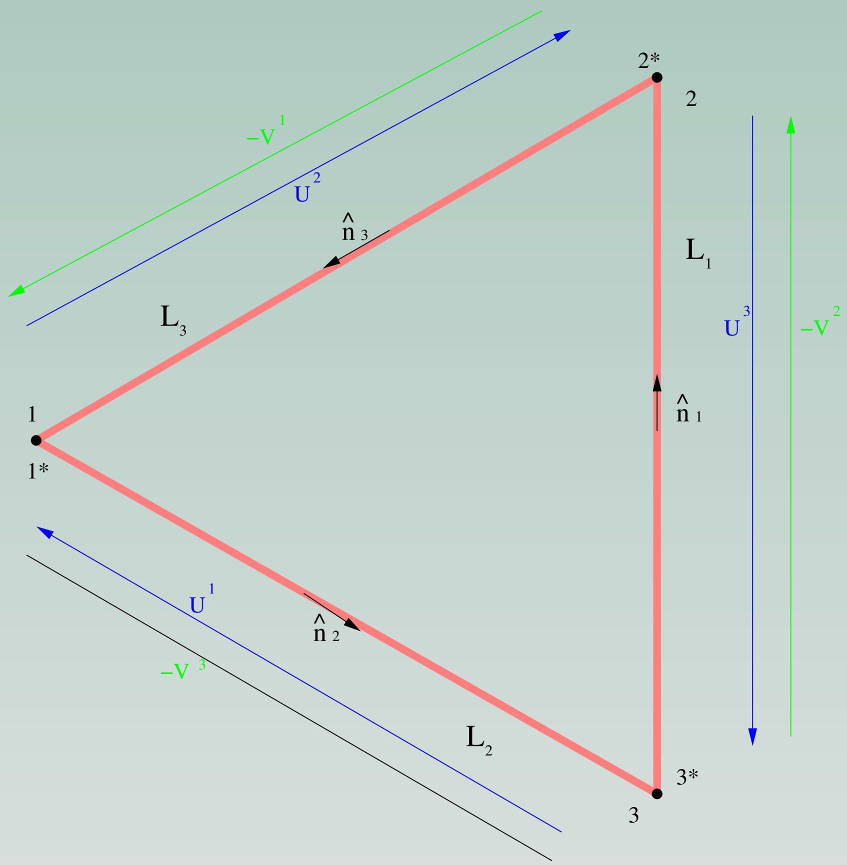



$$\begin{aligned} X(t) &= [C_1(t-2L_2) - C_1(t)] - [C_2(t-2L_1) - C_2(t)] \\ &= [(\mathcal{D}_2^2 - 1)(\mathcal{D}_1^2 - 1) - (\mathcal{D}_1^2 - 1)(\mathcal{D}_2^2 - 1)]C(t) \\ &= 0 \end{aligned}$$

**Just LCM! Commutative algebra! Equal pathlengths**

# Schematic LISA

LISA: Interferometric triangle with unequal arms




- All $L_i$ unequal
- $L_i \sim 5 \times 10^6$ km $\sim 17$ sec.
- Six elementary data streams:

$$U^1, U^2, U^3, V^1, V^2, V^3$$

- Example:

$$U^1 : 3 \longrightarrow 1$$

$$-V^1 : 2 \longrightarrow 1$$

# The six elementary data streams

In LISA the data are recorded as fractional Doppler shifts

Three laser frequency noises: $C_i(t) = \frac{\Delta\nu_i(t)}{\nu_0}$

For example: $U^1(t) = C_3(t - L_2(t)) - C_1(t) + U^1_{\rm GW} + U^1_{\rm opt} + U^1_{\rm pf}$

Algebra of finite difference operators $x, y, z; l, m, n$

$x : 1 \longrightarrow 2, \;\; l : 2 \longrightarrow 1$, ... + cyclic permutations

Displaying only the laser frequency terms: $U^1(t) = zC_3(t) - C_1(t) + \ldots$

$$
\begin{aligned}
U^1 &= zC_3 - C_1 + \ldots \qquad & V^1 &= C_1 - lC_2 + \ldots \\
U^2 &= xC_1 - C_2 + \ldots \qquad & V^2 &= C_2 - mC_3 + \ldots \\
U^3 &= yC_2 - C_3 + \ldots \qquad & V^3 &= C_3 - nC_1 + \ldots
\end{aligned}
$$

# The General TDI for LISA

A general data combination:

$$X = \sum_{i=1}^{3} p_i V^i + q_i U^i$$

where $p_i$ and $q_i$ are the polynomials in $x, y, z, l, m, n$ operators which in general do not commute.

Laser frequency noise terms should vanish for arbitrary $C_i$ if:

$$\begin{aligned} p_1 - q_1 + q_2 x - p_3 n &= 0 \\ p_2 - q_2 + q_3 y - p_1 l &= 0 \\ p_3 - q_3 + q_1 z - p_2 m &= 0 \end{aligned}$$

**Solutions:** Polynomial vectors: **module of syzygies**

# The solution space: A module

Eliminate $p_1$ and $p_2$ by Gaussian elimination:

$$\psi(x,y,z;l,m,n) \equiv p_3(1-nlm)+q_1(z-lm)+q_2(xl-1)m+q_3(ym-1)=0$$

$\mathcal{Q}(x,y,z,l,m,n) \equiv \mathcal{K}$ is in general non-commutative polynomial ring.

Consider the map $\varphi : \mathcal{K}^4 \longrightarrow \mathcal{K}$ which takes the polynomial vector $(p_3, q_1, q_2, q_3) \in \mathcal{K}^4$ to $\psi(x,y,z,l,m,n) \in \mathcal{K}$.

$\varphi$ is a homomorphism of modules and its kernel $\varphi^{-1}(0)$ is essentially the submodule we want.

$\varphi$ can be easily extended to $\mathcal{K}^6$ via the elimination equations and then the kernel of this homomorphism is the module $\mathcal{M} \subset \mathcal{K}^6$.

# Ist and IInd generation TDI

- Stationary LISA in flat spacetime: **Ist generation TDI**
  $x = l, y = m, z = n$ - Operators commute

- TDI needs to be generalised taking into consideration, moving LISA, changing armlengths, gravitational field: A **general relativistic** model of LISA optical links is required to account for Sagnac effect, changing armlengths, gravitational field - Shapiro delay, etc.

- Sagnac Effect - **Modified Ist generation TDI**
  Light travel times in two directions around the Sagnac circuit are different:
  $L_i \neq L'_i, \ \Delta(L_1 + L_2 + L_3) \sim 4A\omega/c \sim 28$ km
  Armlengths constant in time, but $x, y, z, l, m, n$ all different - Operators commute

- Time dependent armlengths - **IInd generation TDI**
  $\dot{L}_i \lesssim 10$ m/sec.
  $\phi(t - L_1 - L_2(t - L_1)) \neq \phi(t - L_2 - L_1(t - L_2))$
  For example: $xy \neq yx$ - Operators do not commute

# Complete solution for Ist generation TDI

Some simple examples found by Estabrook, Tinto, Armstrong, JPL, Pasadena, U.S. - adhoc methods used (1999 - 2001).

Symmetric Sagnac:

$$\zeta = yV^1 + zV^2 + xV^3 + yU^1 + zU^2 + xU^3$$

$$yC_1 - xyC_2 + zC_2 - yzC_3 + xC_3 - zxC_1 + yzC_3 - yC_1 + zxC_1 - zC_2 + xyC_2 - xC_3 = 0$$

This is an identity!

Polynomial vectors in the delay operators:

$$\begin{aligned} \zeta &= (y, z, x, y, z, x) \\ X &= (1 - z^2, 0, z(x^2 - 1), 1 - x^2, x(z^2 - 1), 0) \end{aligned}$$

$X$: Michelson - Sensitivity curve of LISA

General method to generate **ALL** Ist generation TDI

# The General Method

Gaussian elimination of $p_1, p_2$ leads to one equation,

$$(xyz-1)p_3 + (xy-z)q_1 + y(1-x^2)q_2 + (1-y^2)q_3 = 0$$

- Solutions are 4-tuples: polynomial vectors $\{p_3, q_1, q_2, q_3\}$
- Kernel of a module homomorphism
- Solution space: Module - *first module of syzygies*.
- Resubstitution of $p_1$ and $p_2$ - (isomorphic) module of 6-tuple polynomial vectors $\{p_1, p_2, p_3, q_1, q_2, q_3\}$

# Rings and Modules

- Commutative rings with identity - No multiplicative inverse
- Module: Abelian group over a ring
- Ideals: $U$ an ideal in a ring $R$, $u \epsilon U$, $r \epsilon R$ then $ru \epsilon U$
- Polynomial rings: $Q[x], \ Q[x, y, z]$
- Principal ideals, PIDs, *greatest common divisor* - gcd
- Gröbner basis: Generalisation of the gcd in a PID

# Generating the Module of Syzygies

- *Obtain a Gröbner basis for the ideal generated by the coefficients:*
  - $xyz-1, xy-z, y(1-x^2), 1-y^2$
  - Gröbner basis: $\{x^2-1, z^2-1, y-xz\}$
- Gröbner basis to module:
  - Form matrices connecting Gröbner basis to the coefficients and vice-versa, and then the generators of the module are given in terms of the matrices
  - The theorems gaurantee that the full module is generated - independent of the order chosen

The above procedure generates the 4-tuple polynomial generators

The polynomial six tuples $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ can be obtained from the elimination equations.

# Obtaining the generators of the module

Coefficients: $f_1 = xyz - 1$, $f_2 = xz - y$, $f_3 = x(1 - z^2)$, $f_4 = 1 - x^2$ A Gröbner basis for the ideal $U = \{f_1, f_2, f_3, f_4\}$ is:

$$\mathcal{G} = \{g_1 = z^2 - 1, g_2 = y^2 - 1, g_3 = x - yz\}$$

Write: $f_i = d_{ij} g_j$ and $g_j = c_{ji} f_i$, then,

$$d = \begin{pmatrix} 1 & z^2 & yz \\ y & 0 & z \\ -x & 0 & 0 \\ -1 & -z^2 & -(x + yz) \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & -x & z^2 - 1 \\ 1 & -y & 0 & 0 \\ 0 & z & 1 & 0 \end{pmatrix} \quad (1)$$

Rows of I - d.c are one set of generators: A

S-polynomials: Write the S-polynomials in terms of the Gröbner basis $\longrightarrow b$

Rows of $b.c$ are the remaining set of generators $B^\star$

Full set: $AUB^\star$: 7 generators but not independent

# Generators of the Module of Syzygies

Using the software Macaulay 2, we obtain one generating set:

$$
\begin{aligned}
X^{(1)} &= (yx - z, 0, x^2 - 1, 0, zx - y, x^2 - 1) \\
X^{(2)} &= (y, z, x, y, z, x) \\
X^{(3)} &= (1, x, yx, 1, yz, z) \\
X^{(4)} &= (yz, 1, y, x, 1, zx)
\end{aligned}
$$

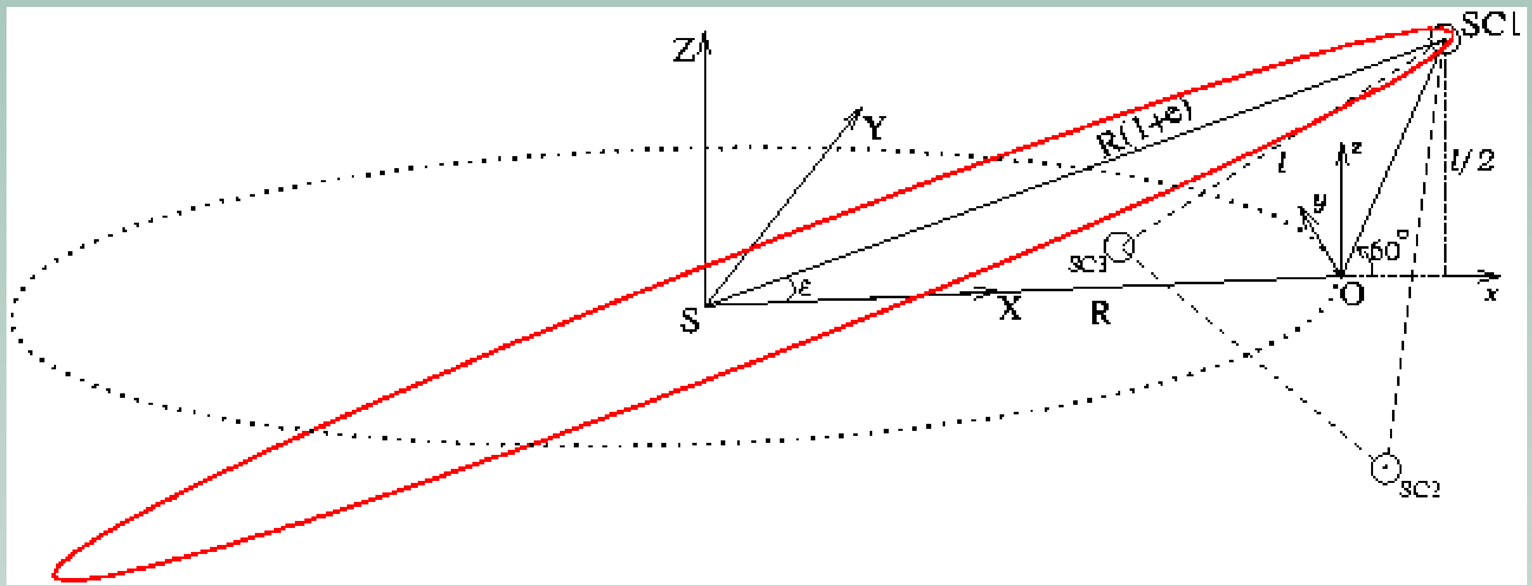
Another generating set:

$$
\begin{aligned}
\alpha &= (1, x, yx, 1, yz, z) \\
\beta &= (yz, 1, y, x, 1, zx) \\
\gamma &= (z, zx, 1, yx, y, 1) \\
\zeta &= (y, z, x, y, z, x)
\end{aligned}
$$

Module is independent of the order chosen for computing the Gröbner basis

# The LISA model in Sun's field

**Minimise flexing:**

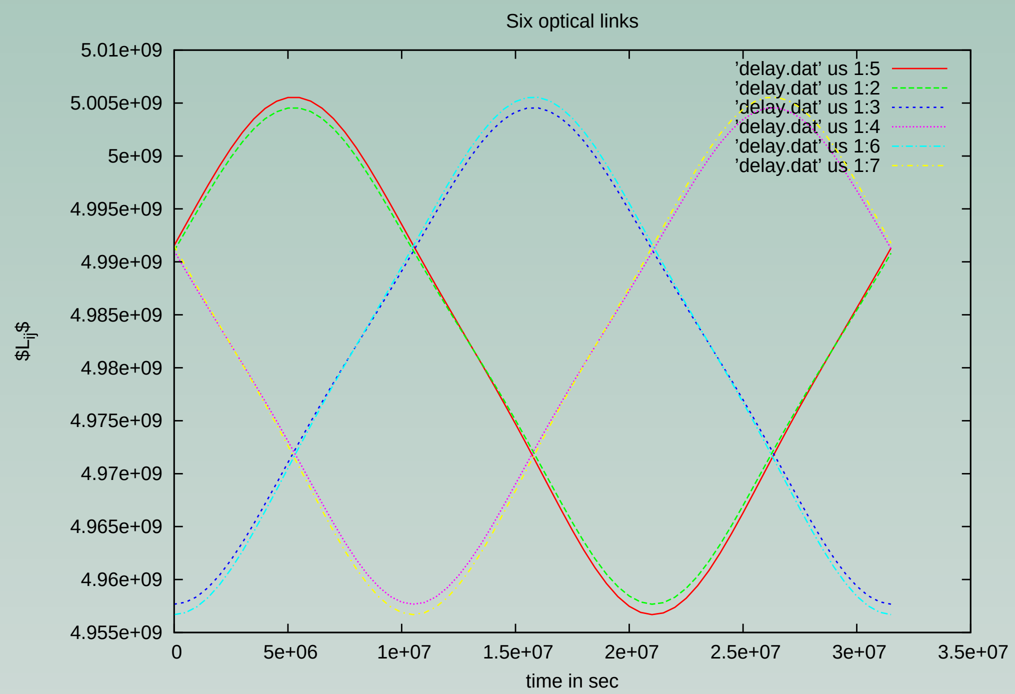



Nayak, Koshti, Dhurandhar, Vinet (2005)

Minimum peak to peak flexing obtained by perturbative analysis of orbital parameters

**Tilt the LISA plane little more by** $5l/16R \sim 0.01$ **radian for minimal flexing**

# Minimal flexing model in Sun's field




Choice of spacecraft orbits: Variation in armlength $\sim 48,000$ km and $\dot{L} \lesssim 4$ m/sec

# Including the Earth: The CW Frame

Transformations to Clohessy-Wiltshire (CW) frame:

$$
\begin{aligned}
x &= (X - R\cos\Omega t)\cos\Omega t + (Y - R\sin\Omega t)\sin\Omega t \\
y &= -(X - R\cos\Omega t)\sin\Omega t + (Y - R\sin\Omega t)\cos\Omega t \\
z &= Z
\end{aligned} \qquad (2)
$$

Coordinates of Earth in CW frame: $(x_\oplus, y_\oplus, 0)$ are constant

$x_\oplus = 9.0 \times 10^6$ km

$y_\oplus = 51.3 \times 10^6$ km

# The perturbation by the Earth

Dhurandhar, Vinet, Nayak (2008)

Perturbed CW equations:

$$\begin{aligned}
\ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x + \epsilon\Omega^2(x - x_\oplus) &= 0\,, \\
\ddot{y} + 2\Omega\dot{x} + \epsilon\Omega^2(y - y_\oplus) &= 0\,, \\
\ddot{z} + \Omega^2(1 + \epsilon)z &= 0
\end{aligned}$$

Ratio of tidal forces of Earth to Sun: $\epsilon = \simeq 7.16 \times 10^{-5}$

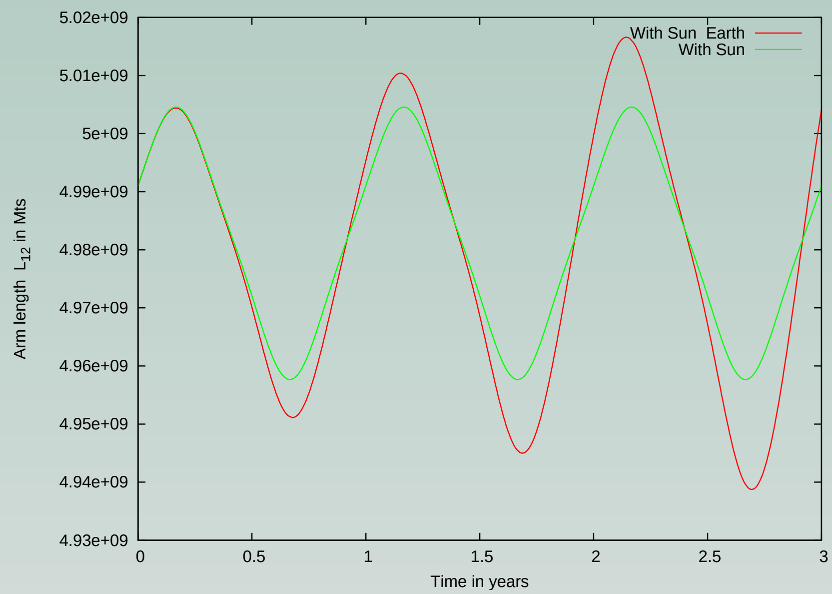
$\epsilon_{\rm Jupiter}/\epsilon_{\rm Earth} \sim 0.09$

$\epsilon_{\rm Moon}/\epsilon_{\rm Earth} \sim 0.01$

# $l_{12}$ in Sun's and Sun + Earth's field

Earth's perturbative effect superposed on the Keplerian S/C orbits:

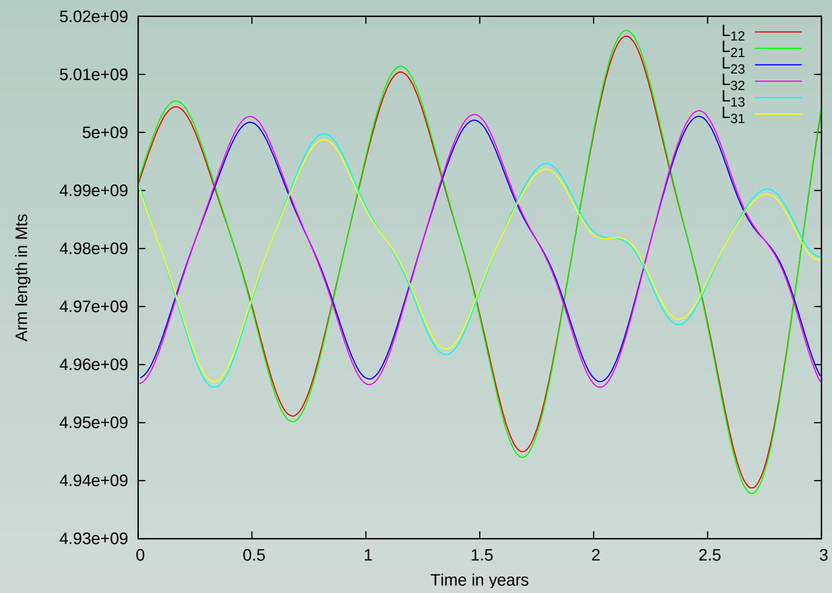
This is done by going to the CW frame, taking a zero'th order solution and perturbing it by the Earth's tidal field




In year 3: Peak increase of $\sim 20,000$ km

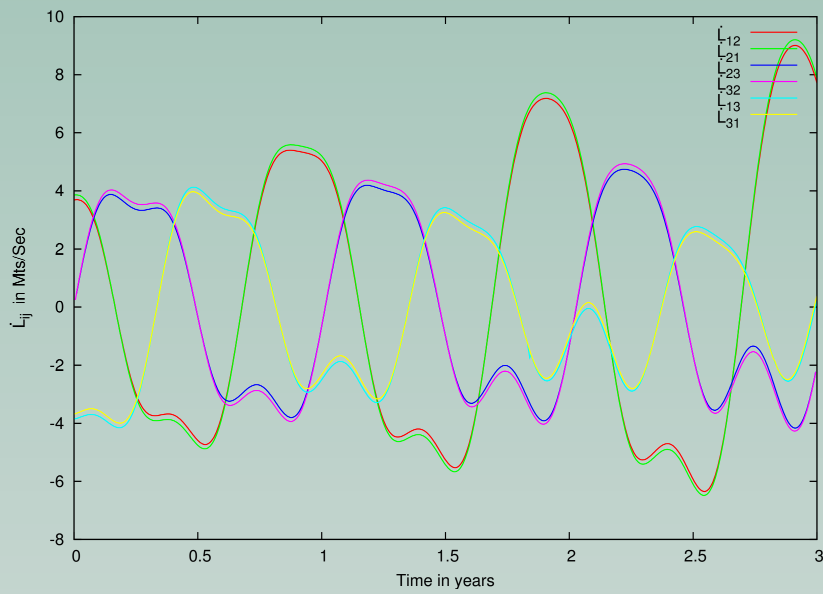
# Optical links: General Relativistic Treatment

- Optical links are obtained by numerically integrating the null geodesics in the Schwarzschild metric determined by Sun's mass.
- Fortan 90 code




Peak to peak variation in the 3rd year $\sim 80,000$ km.

# The $\dot{L}_{ij}$




Year 3: $\dot{L}_{ij} \lesssim 8$ m/sec.

$|\dot{L}_x - \dot{L}_l| \sim 0.1$m/sec, $\quad \frac{1}{c^2}[x, l]C \simeq \Delta t\dot{C} = \Delta C$

$S_{\Delta C}(f = 1\text{mHz}) \sim 10^{-48}$ Hz$^{-1}$ $\longrightarrow x, l$ commute! Approximate symmetry!

# Approximate symmetries

We drop $\ddot{L} \sim 10^{-6}$ m/sec$^2$ and $\dot{L}^2 \sim 10^{-15}$ terms:

$$(jk - kj)C(t) = (L_j \dot{L}_k - L_k \dot{L}_j)\dot{C}(t - L_j - L_k)$$

Since $\dot{L}_x \simeq \dot{L}_l \longrightarrow [x, l] \simeq 0$.

Similarly, $[y, m] \simeq 0, \;\; [z, n] \simeq 0$.

In fact, if $y_1 y_2 \ldots y_n$ is a permutation of $x_1 x_2 \ldots x_n$, where $x_k$ is any of the operators, then,

$$[x_1 x_2 \ldots x_n, y_1 y_2 \ldots y_n] \simeq 0, \quad n \geq 2$$

For example:

$$[xy, yx] = xyyx - yxxy = xy^2x - yx^2y \simeq 0$$

# The quotient ring $\bar{\mathcal{K}}$

Construct the ideal $\mathcal{U} \subset \mathcal{K}$ as follows:

- Use the above nearly vanishing commutators to generate the ideal $\mathcal{U}$ - consider all linear combinations of the commutators.
- Form the quotient ring $\bar{\mathcal{K}} \equiv \mathcal{K}/\mathcal{U}$ - any element of $\bar{p} \in \bar{\mathcal{K}}$ is of the form $\bar{p} = p + \mathcal{U}$, where $p \in \mathcal{K}$ - it is an equivalence class of polynomials in $\mathcal{K}$.
- The new module is now over the quotient ring $\bar{K}$ with each element written as $(\bar{p}_i, \bar{q}_i) \in \bar{\mathcal{K}}^6$

Effectively, there are now less polynomials in $\bar{K}$ than in $\mathcal{K}$ and hence the original ring is simplified; eg. $\bar{l}\bar{x} = \bar{x}\bar{l}$, $\bar{x}\bar{y}^2\bar{x} = \bar{y}\bar{x}^2\bar{y}$, etc.

# Residual Noises in Modified Ist generation TDI

Example of Sagnac $\alpha$:

$\alpha = (\kappa, \kappa l, \kappa lm; \eta, \eta zy, \eta z) \qquad \kappa = 1 - zyx, \eta = 1 - lmn$

$$S_C(f) = \langle |\tilde{C}(f)|^2 \rangle \sim 10^{-27} \quad \mathrm{Hz}^{-1}$$

$$\Delta C = \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3$$

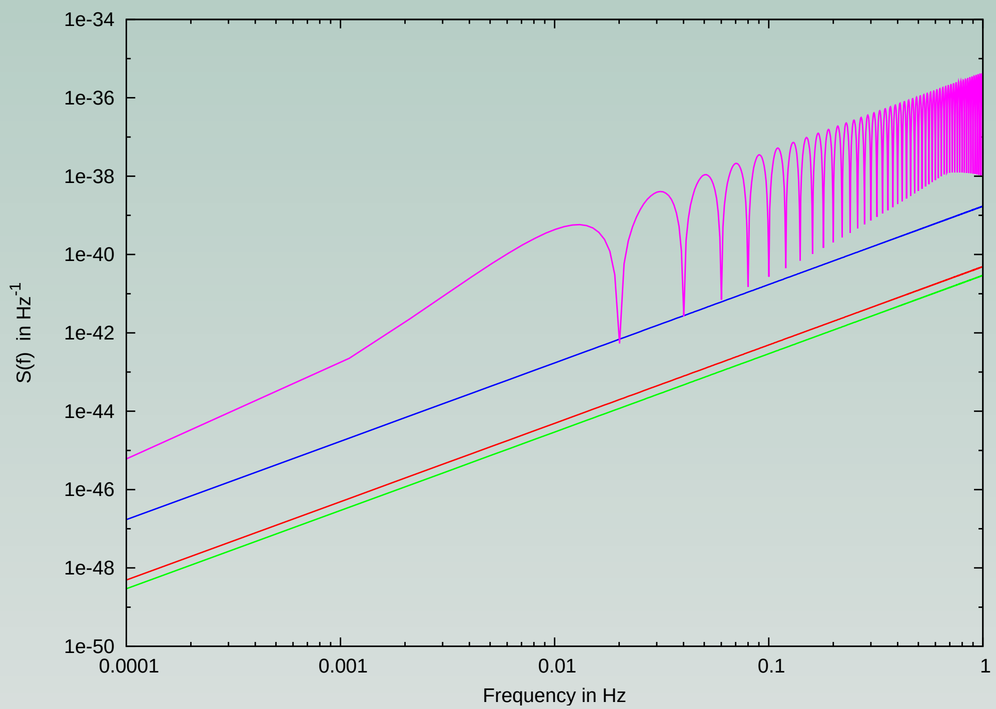
$\alpha_1 = [zyx, lmn], \alpha_2 = \alpha_3 = 0$: Upto first order in $\dot{L}$ and dropping $\ddot{L}$ terms and higher order:

$$\Delta t(t) = \frac{1}{c^2}[(L_x + L_y + L_z)(\dot{L}_l + \dot{L}_m + \dot{L}_n) - (L_l + L_m + L_n)(\dot{L}_x + \dot{L}_y + \dot{L}_z)]$$

# Sagnac

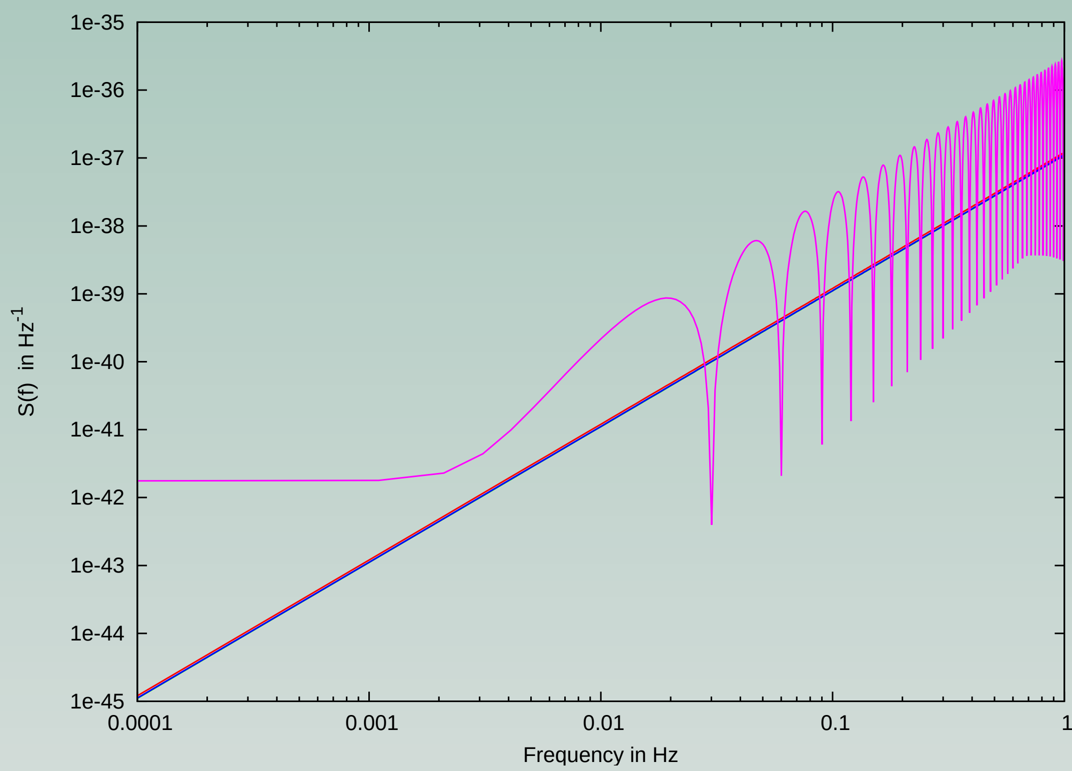
$$S_{\Delta C}(f;t) = 4\pi^2 \Delta t(t)^2 f^2 S_C(f)$$

Also the secondary noise PSD must be multiplied by $4\sin^2(3\pi f L_0)$ because of $\kappa, \eta$



# TDI variable: Michelson

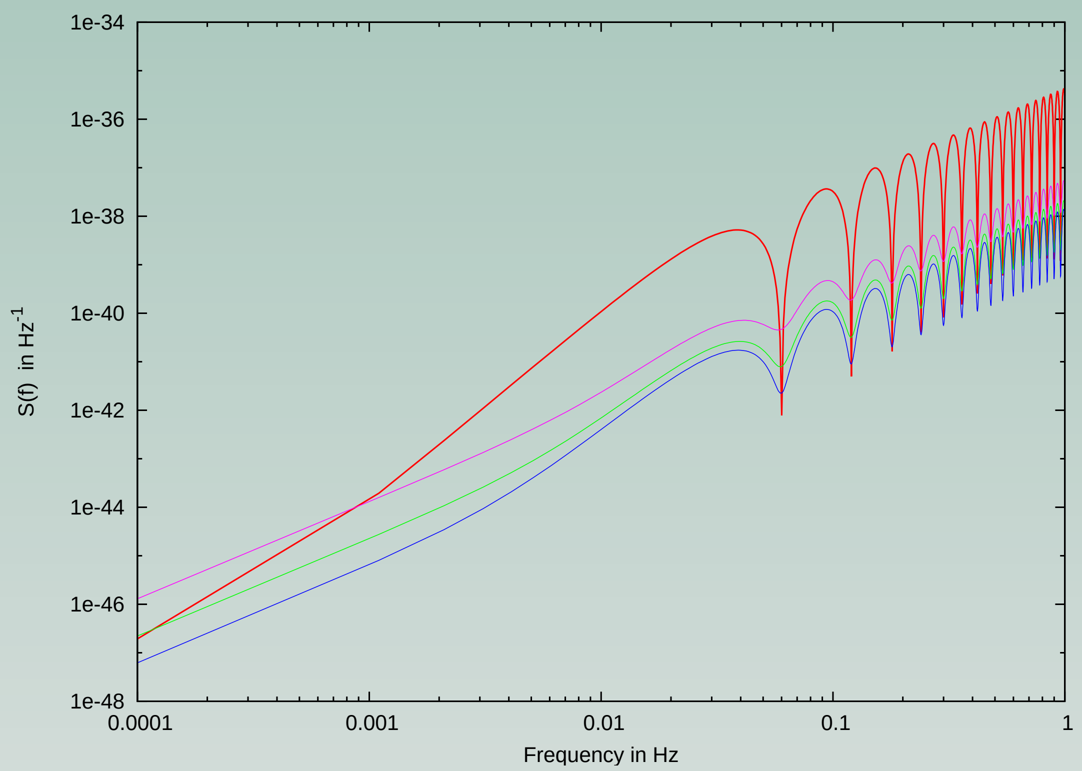
$$X = (1 - zn, 0, (lx - 1)z; 1 - lx, (zn - 1)l, 0)$$




Residual noise about $10 - 20\%$.

# TDI variable: $\zeta$

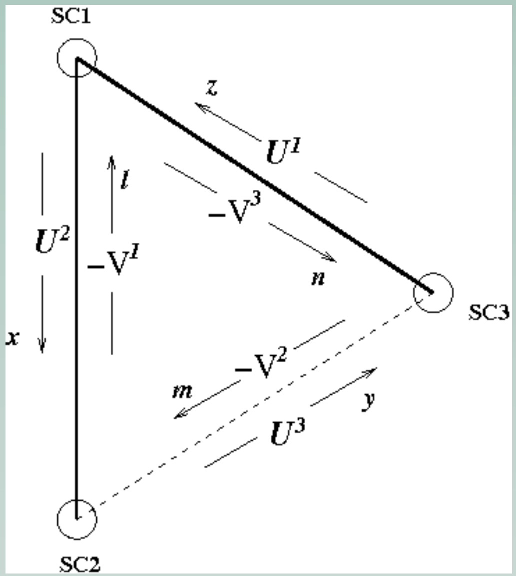
$$\zeta_1 = (y(zx - m), (ln - y)z, (zx - m)l, m(ln - y), (ln - y)z, (zx - m)l)$$




Residual noise suppressed except at low frequencies $\lesssim$ 1 mHz.

# IInd generation TDI for LISA: one arm dysfunctional

The general TDI problem extremely difficult: Special case when one of the arms is dysfunctional, say, connecting S/C 2 to S/C 3.




Links: $x, l, z, n$
Polynomials: $p_2 = q_3 = 0$
Reduced equations: $q_2 = -p_1 l,\ p_3 = -q_1 z$
Only one non-trivial equation: $p_1(1 - lx) - q_1(1 - zn) = 0$

# The Michelson solution: round trip TDI operators

Round trip operators: $a = lx, \;\; b = zn$

Equation: $p_1(1-a) - q_1(1-b) = 0$

Michelson solution: $p_1 = 1 - b - ba + ab^2, \;\; q_1 = 1 - a - ab + ba^2$

Solution because $p_1(1-a) - q_1(1-b) \equiv \Delta = [ba, ab] \simeq 0$
Tinto, Armstrong, Estabrook, Vallisneri

Many more solutions possible of higher degrees

Construct commutators $\in \mathcal{U}$ of higher degrees: To each such commutator there is a solution

# Higher degree solutions $n = 2$

Commutator $\Delta = [ab^2a, ba^2b] \simeq 0$

Solution with this commutator $[ab^2a, ba^2b]$:

$$p_1 = 1 - b - ba + ab^2 - ba^2b + ab^2ab + ab^2aba - ba^2bab^2$$

$$q_1 = 1 - a - ab + ba^2 - ab^2a + ba^2ba + ba^2bab - ab^2aba^2$$

Two other commutators: $[a^2b^2, b^2a^2], [abab, baba] \in \mathcal{U}$

These produce two other linearly independent solutions of degree $7$ in $a, b$ which is $14$ in $x, l, z, n$.

$p_3, q_2$ are of degree $15$.

## Solutions of degree $n \geq 3$

First list the commutators in some order: Choose length lexicographical order $a < b$

$a < b < aa < ab < ba < bb < aaa < aab < aba < abb < baa < bab < bba < bbb < aaaa \ldots.$

A commutator must have equal numbers of $a$'s and $b$'s. List commutators by their first string.

$n = 3$: $aaabbb < aababb < aabbab < aabbba < abaabb \ldots.$ 10 such commutators.

$n = 4$: Length of string 8 in $a, b$ and 35 such commutators

Mathematically: infinite family; Physically restricted by $\ddot{L}$ terms: $n \leq 10$.

# Summary

- Problem solved for Ist generation TDI.
- General relativistic model worked out numerically: Level of 'non-commutativity' involved has been computed.
- Look for an optimal model of LISA which minimises flexing and so that modified Ist generation TDI suffice.
- **OR** use symmetries to simplify the IInd generation TDI problem - problem not fully non-commutative - thus simplify the polynomial ring by quotienting it with an ideal generated by near vanishing commutators.
- Family of solutions: IInd generation TDI for the special case when LISA's one arm is dysfunctional.