ベクトル解析のよく使う公式 電磁気学入門用　改訂版

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1　内積／外積

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C} \tag{1}$$

$$\boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C}) \neq \boldsymbol{B}(\boldsymbol{C} \cdot \boldsymbol{A}) \neq \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{2}$$

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) \tag{3}$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) \tag{4}$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) = 0 \tag{5}$$

2　表面積分についての Gauss の定理

体積 V とその表面 S についての表面積分と体積積分に関して、

$$\mathrm{div}\ \boldsymbol{A} \equiv \lim_{V \to \mathrm{o}} \frac{1}{V} \int_S \boldsymbol{A} \cdot \boldsymbol{n}\, dS \tag{6}$$

$$\int_S \boldsymbol{A} \cdot \boldsymbol{n}\, dS = \int_V \mathrm{div}\ \boldsymbol{A}\, dV \tag{7}$$

3　Green の定理

$$\mathrm{div}(\phi(\boldsymbol{x})\ \boldsymbol{A}(\boldsymbol{x})) = \phi\ \mathrm{div}\boldsymbol{A} + \boldsymbol{A} \cdot \mathrm{grad}\phi \tag{8}$$

$$\mathrm{div\ grad}\ \phi(\boldsymbol{x}) = \triangle\phi \tag{9}$$

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{10}$$

$$\boldsymbol{n}(\boldsymbol{x}) \cdot \mathrm{grad}\ \phi(\boldsymbol{x}) = \boldsymbol{n} \cdot \nabla\phi = \frac{\partial \phi}{\partial n} \tag{11}$$

$$\int_S (\phi(\boldsymbol{x})\ \mathrm{grad}\ \psi(\boldsymbol{x}) - \psi(\boldsymbol{x})\ \mathrm{grad}\ \phi(\boldsymbol{x})) \cdot \boldsymbol{n}\ dS = \int_V (\phi\ \triangle\psi - \psi\ \triangle\phi) dV \tag{12}$$

$$\int_S \left(\phi\ \frac{\partial\psi}{\partial n} - \psi\ \frac{\partial\phi}{\partial n}\right) dS = \int_V (\phi\ \triangle\psi - \psi\ \triangle\phi) dV \tag{13}$$

4　Stokes の定理

閉曲線 r の接線単位ベクトル $\boldsymbol{t}$ に沿った周回積分と、閉曲線に囲まれた曲面 S について、

$$\oint_r \boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{t}\ dr = \tag{14}$$

$$\oint_r \boldsymbol{A}(\boldsymbol{x}) \cdot d\boldsymbol{r} = \int_S (\mathrm{rot}\boldsymbol{A}) \cdot \boldsymbol{n}\ dS \tag{15}$$

5 便利な変形

$$\text{div rot } \boldsymbol{A} = 0 \tag{16}$$

$$\text{grad}(\phi\psi) = \psi \ \text{grad}\phi + \phi \ \text{grad}\psi \tag{17}$$

$$\text{rot}(\phi\boldsymbol{A}) = \phi \ \text{rot}\boldsymbol{A} + \text{grad}\phi \times \boldsymbol{A} \tag{18}$$

$$\text{rot rot}\boldsymbol{A} = \text{grad div}\boldsymbol{A} - \triangle\boldsymbol{A} \tag{19}$$

$$\text{div}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \ \text{rot}\boldsymbol{A} - \boldsymbol{A} \ \text{rot}\boldsymbol{B} \tag{20}$$

$$\boldsymbol{A} \times (\text{rot}\boldsymbol{A}) = \frac{1}{2} \ \text{grad}\boldsymbol{A}^2 - (\boldsymbol{A} \cdot \text{grad})\boldsymbol{A} \tag{21}$$

$$\int_V \text{rot}\boldsymbol{A} \ dV = \int_S (\boldsymbol{n} \times \boldsymbol{A}) dS \tag{22}$$

$$\text{grad}(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \text{grad})\boldsymbol{B} + (\boldsymbol{B} \cdot \text{grad})\boldsymbol{A} + \boldsymbol{A} \times \text{rot}\boldsymbol{B} + \boldsymbol{B} \times \text{rot}\boldsymbol{A} \tag{23}$$

$$\text{rot}(\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A} \ \text{div}\boldsymbol{B} - \boldsymbol{B} \ \text{div}\boldsymbol{A} + (\boldsymbol{B} \cdot \text{grad})\boldsymbol{A} - (\boldsymbol{A} \cdot \text{grad})\boldsymbol{B} \tag{24}$$

$$\int (\text{grad}\phi \times \text{grad}\psi) \cdot \boldsymbol{n} \ dS = \int \phi d\psi \tag{25}$$

$$\oint_r \phi \ d\boldsymbol{r} = \int_S \boldsymbol{n} \times \text{grad}\phi \ dS \tag{26}$$

6　３次元極座標での演算

体積要素

$$dV \;=\; dr\; rd\theta\; r\sin\theta d\phi \tag{27}$$

ラプラシアン

$$\begin{aligned}\triangle u &= \nabla^2 u \\ &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\end{aligned} \tag{28}$$

勾配

$$\mathrm{grad}\; u \;=\; \frac{\partial u}{\partial r}\boldsymbol{e_r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\boldsymbol{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\boldsymbol{e_\phi} \tag{29}$$

湧き出し

$$\begin{aligned}\mathrm{div}\;\boldsymbol{A} &= \nabla\cdot\boldsymbol{A} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(A_\phi\right)\end{aligned} \tag{30}$$

回転

$$\begin{aligned}\mathrm{rot}\;\boldsymbol{A} &= \nabla\times\boldsymbol{A} \\ &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right)\boldsymbol{e_r} \\ &\quad + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\boldsymbol{e_\theta} \\ &\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\boldsymbol{e_\phi}\end{aligned} \tag{31}$$

速度

$$\frac{d\boldsymbol{r}}{dt} \;=\; \frac{dr}{dt}\boldsymbol{e_r} + r\frac{d\theta}{dt}\boldsymbol{e_\theta} + r\sin\theta\frac{d\phi}{dt}\boldsymbol{e_\phi} \tag{32}$$

加速度

$$\begin{aligned}\frac{d^2\boldsymbol{r}}{dt^2} &= \left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right)\boldsymbol{e_r} \\ &\quad + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\right)\boldsymbol{e_\theta} \\ &\quad + \left(\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\;\frac{d\phi}{dt}\right)\right)\boldsymbol{e_\phi}\end{aligned} \tag{33}$$

最新版は以下の URL：

`http://www.gw.hep.osaka-cu.ac.jp/4students/`